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SOME THEOREMS IN THE THEORY OF SMALL DEFORMATIONS SUPERIMPOSED ON A FINITE DEFORMATION OF A HYPERELASTIC MATERIAL OF GRADE 2

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Abstract—The equations for small deformations superimposed on large deformations of a hyperelastic material of grade 2 are formulated and applied to derive a basic integral relation that is used to establish generalized Betti, Clapeyron, work and energy theorems. Theorems of minimum and complementary energy are deduced essentially from an energy criterion of super-stability, and these are used to prove uniqueness of solutions to the static and dynamic, mixed incremental boundary value problems. These results use a certain generalized kinetic energy functional that is assumed positive definite; and this property and the reciprocal and energy principles are exploited further to establish some theorems in the theory of small free vibrations.

1. INTRODUCTION

The finite theory of hyperelasticity for Cosserat and grade 2 materials in equilibrium was developed by Toupin[1] in 1962. Two years later he developed in [2] a more comprehensive and deep, analysis of oriented hyperelastic materials characterized by a set of deformable directors, and he reformulated the grade 2 theory on the basis of an elegant, though possibly deceptively simple, variational equation. However, the two theories are developed along quite different directions, and there is no mention of any connection between the director-oriented and non-simple, grade 2 continua. We have shown in [3] that an oriented continuum in which the director triad is constrained to stretch and rotate in harmony with the local deformation of material line elements of the continuum is a grade 2 material; of course, not every grade 2 material need be characterized in this manner. Nevertheless, we have derived from Toupin's director theory the variational principle and basic general equations for hyperelastic materials of grade 2. Though equivalent to the theory proposed by Toupin[2], these equations have a different structure that is easier to use and interpret in more direct physical terms. These results are sketched in Section 2.

Toupin[1] has described some features of plane wave propagation in the context of the theory of small vibrations superimposed on an equilibrium configuration of a hyperelastic Cosserat continuum in homogeneous, finite strain without body or hyper-body forces and having only the usual linear momentum. Toupin and Gazis[4] have applied the incremental theory to study a uniform crystal in which initial stress and hyperstress produced surface deformations in a thin boundary layer characteristic of deformations reported in electron diffraction experiments. In the present study, we shall apply this theory in the proofs of several fundamental theorems for general hyperelastic grade 2 materials. The equations for the theory of small deformations superimposed on large deformations of a hyperelastic material of grade 2 are derived in Section 3 from the theory formulated in [3].

The total incremental strain energy and generalized kinetic energy are defined in Section 4, and in Section 5 we derive a useful integral relation that yields as special cases generalized versions of the classical Betti and Clapeyron theorems and a basic energy theorem. Similar theorems of the Betti–Clapeyron type have been proved by Beatty[5] for hyperelastic materials in the linearized theory of couple-stresses, and, about the same time, also by Sandru[6] for the particular isotropic material; but the energy theorem, which includes the classical result, was not considered. Moreover, the present more general analysis is significantly different and more complex in its structure as to warrant separate attention for itself.

A dead load stability criterion and related definitions of D_L stability and superstability for grade 2 materials are provided in Section 6. It is shown that the linearized dead load criterion is equivalent to the criterion that the second variation of a certain potential energy functional be non-negative for all infinitesimal virtual deformations that satisfy the boundary conditions. The result is based upon the variational equation derived in [3].

In Section 7, theorems of minimum and maximum potential energy are deduced essentially from the criterion of superstability. Parallel theorems for simple hyperelastic materials have been derived by Shield and Fosdick [7]; their results are here included as a special case. With the aid of these principles, which are similar to the classical theorems, it is shown that the solution of the mixed, incremental boundary value problem of equilibrium is unique; the traction problem solution, as usual, is unique only to within a rigid motion. In addition, it is shown in Section 8 that the weaker D_L stability criterion is sufficient for uniqueness of the corresponding dynamic problem provided that the generalized kinetic energy functional is positive definite, as it is in the classical case to which the general definition reduces for special circumstances.

Normal mode functions are introduced in Section 9 and from our reciprocal theorem we find, subject to certain boundary conditions, that these functions satisfy a certain generalized orthogonality condition that includes as an example the well known classical criterion for orthogonal vector functions. Finally, we prove that if the kinetic energy is a positive definite functional and the underlying equilibrium configuration is D_L superstable and certain boundary conditions can be met, then in a superimposed infinitesimal vibration all frequencies are real and positive, which is an easy consequence of our energy theorem.

2. BASIC EQUATIONS FOR MATERIALS OF GRADE 2

The field equations and natural boundary conditions for a hyperelastic grade 2 material with response function

$$L = L(X, \dot{\mathbf{x}}, \mathbf{F}, \dot{\mathbf{F}}, \mathbf{G}, t), \tag{2.1}$$

per unit volume in an assigned reference configuration κ, are given by

$$\operatorname{Div}\hat{\mathbf{T}} + \mathbf{Z} = \dot{\mathbf{P}} \quad \text{for all } X \in \mathfrak{P}, \tag{2.2}$$

$$\mathbf{T}_N = \hat{\mathbf{T}}\mathbf{N} + \mathbf{D}\cdot\boldsymbol{\Phi}, \quad \boldsymbol{\Phi}\mathbf{N} = \mathbf{0} \quad \text{for all } X \in \partial\mathfrak{P}, \tag{2.3}$$

$$[\boldsymbol{\Phi}\mathbf{M}] = \mathbf{0} \quad \text{for all } X \in \mathscr{C}, \tag{2.4}$$

wherein, by definition,

$$\hat{\mathbf{T}} \equiv \mathbf{T} - \mathbf{K} + \dot{\mathbf{Q}}, \quad \boldsymbol{\Phi} \equiv \mathbf{H}_N - \mathbf{H}\mathbf{N}. \tag{2.5}$$

In these relations, which are derived in [3], the momentum vector $\mathbf{P}$, the hypermomentum tensor $\mathbf{Q}$, the first Piola-Kirchhoff stress tensor $\mathbf{T}$ and the hyperstress tensor $\mathbf{H}$, which are referred to the boundary $\partial\mathfrak{P}$ of the body $\mathfrak{P}$ in κ, are determined by (2.1) in accordance with

$$\mathbf{T} = -\left(\frac{\partial L}{\partial \mathbf{F}} + \operatorname{Div}\mathbf{H}\right), \quad \mathbf{H} = -\frac{\partial L}{\partial \mathbf{G}}, \quad \mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{x}}}, \quad \mathbf{Q} = \frac{\partial L}{\partial \dot{\mathbf{F}}}. \tag{2.6}$$

In addition, $\mathbf{T}_N$, $\mathbf{H}_N$ define the traction vector and hypertraction tensor referred to κ wherein $\mathbf{X}$ is the place occupied by the particle X whose position vector and velocity in the present configuration χ at time t are $\mathbf{x}(\mathbf{X}, t)$ and $\dot{\mathbf{x}}(\mathbf{X}, t)$, respectively; $\mathbf{Z}$ and $\mathbf{K}$ denote the body force and hyper-body force per unit volume in κ; and $\mathbf{G} = \nabla\mathbf{F}$ is the material gradient of the deformation gradient $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$. Also, a superimposed dot denotes the material time derivative; the divergence operator refers to material variables; $\mathbf{D}$ is the material surface gradient, and its inner product with any tensor $\mathbf{V}$ defines a surface divergence $D_\gamma V^{\alpha\cdots\beta\gamma}$. Finally, the bold brackets denote the jump in the enclosed quantity as the boundary edges $\mathscr{C}$ are approached from each side, and $\mathbf{M} \equiv \mathbf{S} \times \mathbf{N}$, where $\mathbf{S}$ is the usual left oriented unit tangent vector to $\mathscr{C}$ in κ and $\mathbf{N}$ is the unit normal vector to $\partial\mathfrak{P}$ in κ.

3. INCREMENTAL EQUATIONS FOR GRADE 2 MATERIALS

Let $\chi'(\mathbf{X}, t)$ and $\chi(\mathbf{X}, t)$ be two smooth motions of $\mathfrak{B}$ referred to κ. Then the superimposed incremental motion $\mathbf{u}(\mathbf{X}, t)$ and its first two material gradients are defined by

$$\mathbf{u}(\mathbf{X}, t) = \chi'(\mathbf{X}, t) - \chi(\mathbf{X}, t), \quad \mathbf{f} = \nabla\mathbf{u} = \partial\mathbf{u}/\partial\mathbf{X}, \quad \mathbf{g} = \nabla\mathbf{f} = \overset{(2)}{\nabla}\mathbf{u}. \tag{3.1}$$

We shall require that the magnitudes of these quantities and their material time derivatives be sufficiently small compared to unity so that subsequent linearization in terms of these is rendered meaningful.

Let a primed entity, say $\mathbf{V}'$, bear the same interpretation with respect to the motion $\chi'(\mathbf{X}, t)$ as the corresponding unprimed entity has with respect to $\chi(\mathbf{X}, t)$. Then for any tensor $\mathbf{V}$ we write $\mathbf{V}^* \equiv \mathbf{V}' - \mathbf{V}$ for the increment $\mathbf{V}^*$ of $\mathbf{V}$. Moreover, we shall consider only hyperelastic grade 2 materials for which the response function (2.1) has the separable form

$$L = \eta(\mathbf{X}, \dot{\mathbf{x}}, \dot{\mathbf{F}}) - \Sigma(\mathbf{x}, \mathbf{F}, \mathbf{G}), \tag{3.2}$$

in which η is identified as the *kinetic energy density* and Σ is the *elastic strain energy density*, both taken per unit volume in κ. We shall suppose that both functions are continuously differentiable in each of their respective arguments. Thus, substituting (3.2) into (2.6) and expanding the results in a Taylor series to first order in $\mathbf{f}$, $\mathbf{g}$, $\dot{\mathbf{f}}$, $\dot{\mathbf{g}}$ defined by (3.1), we find that the incremental stress $\mathbf{T}^*$, hyperstress $\mathbf{H}^*$, momentum $\mathbf{P}^*$ and hypermomentum Q^* are determined by†

$$\mathbf{T}^* = \mathbf{A} \cdot \mathbf{f} + \mathbf{B} \cdot \mathbf{g} - \operatorname{Div} \mathbf{H}^*, \quad \mathbf{H}^* = \mathbf{B}^T \cdot \mathbf{f} + \mathbf{C} \cdot \mathbf{g}, \tag{3.3}$$

$$\mathbf{P}^* = \boldsymbol{\alpha} \cdot \dot{\mathbf{u}} + \boldsymbol{\beta} \cdot \dot{\mathbf{f}}, \quad \mathbf{Q}^* = \boldsymbol{\beta}^T \cdot \dot{\mathbf{u}} + \boldsymbol{\gamma} \cdot \dot{\mathbf{f}}, \tag{3.4}$$

wherein, by definition,

$$\mathbf{A} = \frac{\partial^2\Sigma}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \mathbf{B} = \frac{\partial}{\partial\mathbf{G}}\left(\frac{\partial\Sigma}{\partial\mathbf{F}}\right), \quad \mathbf{B}^T = \frac{\partial}{\partial\mathbf{F}}\left(\frac{\partial\Sigma}{\partial\mathbf{G}}\right), \quad \mathbf{C} = \frac{\partial^2\Sigma}{\partial\mathbf{G}\partial\mathbf{G}}, \tag{3.5}$$

$$\boldsymbol{\alpha} = \frac{\partial^2\eta}{\partial\dot{\mathbf{x}}\partial\dot{\mathbf{x}}}, \quad \boldsymbol{\beta} = \frac{\partial}{\partial\dot{\mathbf{F}}}\left(\frac{\partial\eta}{\partial\dot{\mathbf{x}}}\right), \quad \boldsymbol{\beta}^T = \frac{\partial}{\partial\dot{\mathbf{x}}}\left(\frac{\partial\eta}{\partial\dot{\mathbf{F}}}\right), \quad \boldsymbol{\gamma} = \frac{\partial^2\eta}{\partial\dot{\mathbf{F}}\partial\dot{\mathbf{F}}}, \tag{3.6}$$

all of which are to be evaluated for $\chi(\mathbf{X}, t)$. In particular, when the kinetic energy density is given by

$$\eta = \tfrac{1}{2}\rho_\kappa \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \tfrac{1}{2}\mathbf{I}\dot{\mathbf{F}} \cdot \dot{\mathbf{F}}, \tag{3.7}$$

in which ρ_κ is the mass density in κ and $\mathbf{I}$ is a constant, positive symmetric tensor, (3.6) gives $\boldsymbol{\alpha} = \rho_\kappa \mathbf{1}$, $\boldsymbol{\beta} = \boldsymbol{\beta}^T = \mathbf{0}$ and $\boldsymbol{\gamma} = I^{\iota\kappa}\delta^{\gamma\beta}\mathbf{e}_{\iota\gamma\kappa\beta}$. Hence, (3.4) yields, for example, $\mathbf{P}^* = \rho_\kappa \dot{\mathbf{u}}$, $\mathbf{Q}^* = \mathbf{I}\dot{\mathbf{f}}$.

Equations (3.3)–(3.4) are the constitutive equations for infinitesimal deformations superimposed on an initial deformation of a hyperelastic grade 2 material. The equations of balance for the superimposed motion are obtained easily from (2.2)–(2.5):

$$\operatorname{Div} \hat{\mathbf{T}}^* + \mathbf{Z}^* = \dot{\mathbf{P}}^*, \tag{3.8}$$

$$\mathbf{T}^*_N = \hat{\mathbf{T}}^*\mathbf{N} + \mathbf{D} \cdot \boldsymbol{\Phi}^*, \quad \boldsymbol{\Phi}^*\mathbf{N} = \mathbf{0} \text{ on } \partial\mathfrak{P}; \quad [\boldsymbol{\Phi}^*\mathbf{M}] = \mathbf{0} \text{ on } \mathscr{C}, \tag{3.9}$$

where

$$\hat{\mathbf{T}}^* = \mathbf{T}^* - \mathbf{K}^* + \dot{\mathbf{Q}}^*, \quad \boldsymbol{\Phi}^* = \mathbf{H}^*_N - \mathbf{H}^*\mathbf{N}. \tag{3.10}$$

†The inner product of a tensor $\mathbf{U}$ of rank $p+q$ with a tensor $\mathbf{V}$ of rank q is defined in the tensor basis $\mathbf{e}_{\mu\ldots\nu} = \mathbf{e}_\mu \otimes \cdots \otimes \mathbf{e}_\nu$ by

$$\mathbf{U} \cdot \mathbf{V} = U^{\alpha\ldots\beta\gamma\ldots\delta} V_{\gamma\ldots\delta} \mathbf{e}_{\alpha\ldots\beta}.$$

4. INCREMENTAL STRAIN ENERGY AND KINETIC ENERGY

Let **a**, **b**, **c** be any twice differentiable vector fields and define the inner product†

$$\langle \mathbf{a}, \mathbf{b}\rangle \equiv \tfrac{1}{2}\{\mathbf{A}[\nabla \mathbf{a}, \nabla \mathbf{b}] + \mathbf{B}[\nabla \mathbf{a}, \overset{(2)}{\nabla} \mathbf{b}] + \mathbf{B}^T[\overset{(2)}{\nabla} \mathbf{a}, \nabla \mathbf{b}] + \mathbf{C}[\overset{(2)}{\nabla} \mathbf{a}, \overset{(2)}{\nabla} \mathbf{b}]\}. \tag{4.1}$$

In view of (3.5), (4.1) has the commutative and distributive properties

$$\langle \mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{b}, \mathbf{a}\rangle, \quad \langle l\mathbf{a} + m\mathbf{b}, \mathbf{c}\rangle = l\langle \mathbf{a}, \mathbf{c}\rangle + m\langle \mathbf{b}, \mathbf{c}\rangle, \tag{4.2}$$

in which l, m are scalars. The usual rule for the derivative of a product also holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, \dot{\mathbf{b}}\rangle + \langle \dot{\mathbf{a}}, \mathbf{b}\rangle. \tag{4.3}$$

The product (4.1) holds particular interest when $\mathbf{a} = \mathbf{b} = \mathbf{u}$. Expansion of the strain energy in a Taylor series to second order in **f** and **g** and use of (2.6) reveals that the incremental strain energy Σ^* is given by

$$\Sigma^* = (\mathbf{T} + \mathrm{Div}\,\mathbf{H})\cdot \mathbf{f} + \mathbf{H}\cdot \mathbf{g} + \langle \mathbf{u}, \mathbf{u}\rangle; \tag{4.4}$$

hence, $\langle \mathbf{u}, \mathbf{u}\rangle$ is the second order increment in the incremental strain energy density, which with the help of (3.3) and (3.5) can be written

$$2\langle \mathbf{u}, \mathbf{u}\rangle = (T^* + \mathrm{Div}\,\mathbf{H}^*)\cdot \mathbf{f} + \mathbf{H}^*\cdot \mathbf{g}. \tag{4.5}$$

Thus, the second order contribution to the total strain energy increment for the infinitesimal superimposed motion (3.1) is given by

$$U \equiv \int_{\mathfrak{P}} \langle \mathbf{u}, \mathbf{u}\rangle\,\mathrm{d}V = \int_{\mathfrak{P}} \tfrac{1}{2}\{\mathbf{A}[\mathbf{f}, \mathbf{f}] + \mathbf{B}[\mathbf{f}, \mathbf{g}] + \mathbf{B}^T[\mathbf{g}, \mathbf{f}] + \mathbf{C}[\mathbf{g}, \mathbf{g}]\}\,\mathrm{d}V. \tag{4.6}$$

Similarly, let **c**, **d** be any differentiable vector fields and define the inner product

$$\langle\langle \mathbf{c}, \mathbf{d}\rangle\rangle \equiv \tfrac{1}{2}\{\boldsymbol{\alpha}[\mathbf{c}, \mathbf{d}] + \boldsymbol{\beta}[\mathbf{c}, \nabla \mathbf{d}] + \boldsymbol{\beta}^T[\nabla \mathbf{c}, \mathbf{d}] + \boldsymbol{\gamma}[\nabla \mathbf{c}, \nabla \mathbf{d}]\}. \tag{4.7}$$

It is clear that (3.6) implies for (4.7) the same kind of commutative, distributive and derivative properties as in (4.2)–(4.3). With the aid of (3.4) and (3.6), (4.7) can also be written

$$\langle\langle \mathbf{c}, \mathbf{d}\rangle\rangle = \tfrac{1}{2}[\mathbf{P}^*(\mathbf{d})\cdot \mathbf{c} + \mathbf{Q}^*(\mathbf{d})\cdot \nabla \mathbf{c}]. \tag{4.8}$$

Upon expansion of the kinetic energy function to terms of second order in $\dot{\mathbf{u}}$ and $\dot{\mathbf{f}}$, and use of (4.7) with $\mathbf{c} = \mathbf{d} = \dot{\mathbf{u}}$ shows that $\langle\langle \dot{\mathbf{u}}, \dot{\mathbf{u}}\rangle\rangle$ is the second order increment in η^*. Hence, the second order increment in the total incremental kinetic energy is given by

$$K = \int_{\mathfrak{P}} \langle\langle \dot{\mathbf{u}}, \dot{\mathbf{u}}\rangle\rangle\,\mathrm{d}V = \int_{\mathfrak{P}} \tfrac{1}{2}\{\boldsymbol{\alpha}[\dot{\mathbf{u}}, \dot{\mathbf{u}}] + \boldsymbol{\beta}[\dot{\mathbf{u}}, \dot{\mathbf{f}}] + \boldsymbol{\beta}^T[\dot{\mathbf{f}}, \dot{\mathbf{u}}] + \boldsymbol{\gamma}[\dot{\mathbf{f}}, \dot{\mathbf{f}}]\}\,\mathrm{d}V. \tag{4.9}$$

5. THE RECIPROCAL, WORK AND ENERGY THEOREMS

Let $\mathbf{v}(\mathbf{X}, t)$ be a twice continuously differentiable vector field on $\mathfrak{P}$ and $\partial\mathfrak{P}$, and let us consider the integral

$$I = \int_{\partial\mathfrak{P}} (\mathbf{T}_N^*\cdot \mathbf{v} + \mathbf{H}_N^*\cdot \nabla \mathbf{v})\,\mathrm{d}A + \int_{\mathfrak{P}} (\mathbf{Z}^*\cdot \mathbf{v} + \mathbf{K}^*\cdot \nabla \mathbf{v})\,\mathrm{d}V - \int_{\mathfrak{P}} (\dot{\mathbf{P}}^*\cdot \mathbf{v} + \dot{\mathbf{Q}}^*\cdot \nabla \mathbf{v})\,\mathrm{d}V. \tag{5.1}$$

†If **U** is a $p+q$-tensor, **V** is a p-tensor and **W** is a q-tensor, then $\mathbf{U}[\mathbf{V}, \mathbf{W}] \equiv \mathbf{V}\cdot(\mathbf{U}\cdot\mathbf{W}) = U^{\alpha\ \ \beta\gamma\cdot\delta}V_{\alpha\cdot\beta}W_{\gamma\cdot\cdot\delta}$ is a scalar product.

Using $(3.9)_{1,2}$ and (3.10), and noting that on $\partial\mathfrak{P}$ in κ

$$\nabla \mathbf{v} = \mathbf{D}\mathbf{v} + (D\mathbf{v}) \otimes \mathbf{N}, \quad \mathbf{D}(\tfrac{1}{2}\mathbf{N}\cdot\mathbf{N}) = -\hat{\mathbf{B}}\mathbf{N} = \mathbf{0}, \tag{5.2}$$

wherein $\hat{\mathbf{B}} = -\mathbf{D}\mathbf{N}$ is the symmetric second fundamental form and D is the normal derivative for $\partial\mathfrak{P}$ [see Ref. 8], we find

$$\int_{\partial\mathfrak{P}} (\mathbf{T}^*_N \cdot \mathbf{v} + \mathbf{H}^*_N \cdot \nabla\mathbf{v})\, dA = \int_{\partial\mathfrak{P}} (\mathbf{v}\cdot\hat{\mathbf{T}}^*\mathbf{N} + \nabla\mathbf{v}\cdot\mathbf{H}^*\mathbf{N})\, dA - \int_{\partial\mathfrak{P}} \{\mathbf{D}\cdot(\mathbf{N}\otimes\mathbf{v}\mathbf{\Phi}^*)\}\cdot\mathbf{N}\, dA.$$

However, application of Toupin's integral identity[1, 9] to the last integral and use of the boundary relations $(3.9)_{2,3}$ reveals that this term is zero; therefore, with the divergence theorem and the incremental balance equation (3.8), (5.1) can be written as

$$I = \int_{\mathfrak{P}} [(\mathbf{T}^* + \operatorname{Div}\mathbf{H}^*)\cdot\nabla\mathbf{v} + \mathbf{H}^*\cdot\overset{(2)}{\nabla}\mathbf{v}]\, dV. \tag{5.3}$$

Finally, with the aid of (3.3), (4.1), (4.2) and (5.1), (5.3) yields

$$I = 2\int_{\mathfrak{P}} \langle\mathbf{v},\mathbf{u}\rangle\, dV = 2\int_{\mathfrak{P}} \langle\mathbf{u},\mathbf{v}\rangle\, dV = \int_{\partial\mathfrak{P}} (\mathbf{T}^*_N\cdot\mathbf{v} + \mathbf{H}^*_N\cdot\nabla\mathbf{v})\, dA$$
$$+ \int_{\mathfrak{P}} (\mathbf{Z}^*\cdot\mathbf{v} + \mathbf{K}^*\cdot\nabla\mathbf{v})\, dV - \int_{\mathfrak{P}} (\dot{\mathbf{P}}^*\cdot\mathbf{v} + \dot{\mathbf{Q}}^*\cdot\nabla\mathbf{v})\, dV \tag{5.4}$$

for all $\mathbf{u}$ (or $\mathbf{v}$) that satisfy (3.8)–(3.9) for assigned $\mathbf{Z}^*$ and $\mathbf{K}^*$.

Beatty[5] has remarked that the classical Betti reciprocal theorem and the Clapeyron work theorem hold within the more general context of the theory of small deformations superimposed on large deformations of hyperelastic materials with couple-stresses. On the basis of the general integral relation (5.4) we can now prove easily for more general grade 2 materials the theorems suggested in [5].

Let $\mathbf{u}(\mathbf{X}, t)$ and $\bar{\mathbf{u}}(\mathbf{X}, t)$ be two incremental motions of a hyperelastic body $\mathfrak{P}$ of grade 2, and let $A = \{\mathbf{T}^*_N, \mathbf{H}^*_N, \mathbf{Z}^*, \mathbf{K}^*, -\dot{\mathbf{P}}^*, -\dot{\mathbf{Q}}^*\}$ and $\bar{A} = \{\bar{\mathbf{T}}^*_N, \bar{\mathbf{H}}^*_N, \bar{\mathbf{Z}}^*, \bar{\mathbf{K}}^*, -\dot{\bar{\mathbf{P}}}^*, -\dot{\bar{\mathbf{Q}}}^*\}$ represent, with respect to κ, two sets of incremental actions consisting of surface tractions and hypertractions per unit area, body forces and hyper-body forces per unit volume, inertia forces and hyper-inertia forces per unit volume, that are necessary to produce the two motions $\mathbf{u}, \bar{\mathbf{u}}$, respectively; and let $D = \{\mathbf{u}, \mathbf{f}\}$ and $\bar{D} = \{\bar{\mathbf{u}}, \bar{\mathbf{f}}\}$ define two corresponding sets of generalized displacements for these [see (3.1)]. Then putting $\mathbf{v} = \bar{\mathbf{u}}$ in (5.4), we have at once the following

Reciprocal Theorem: *The total work that would be done on a hyperelastic body $\mathfrak{P}$ of grade 2 by the actions A acting over the generalized displacements $\bar{D}$ produced by the actions $\bar{A}$ is equal to the total work that would be done on $\mathfrak{P}$ by the actions $\bar{A}$ acting over the generalized displacements D produced by the actions A:*

$$\int_{\partial\mathfrak{P}} (\mathbf{T}^*_N\cdot\bar{\mathbf{u}} + \mathbf{H}^*_N\cdot\bar{\mathbf{f}})\, dA + \int_{\mathfrak{P}} (\mathbf{Z}^*\cdot\bar{\mathbf{u}} + \mathbf{K}^*\cdot\bar{\mathbf{f}})\, dV - \int_{\mathfrak{P}} (\dot{\mathbf{P}}^*\cdot\bar{\mathbf{u}} + \dot{\mathbf{Q}}^*\cdot\bar{\mathbf{f}})\, dV$$
$$= \int_{\partial\mathfrak{P}} (\bar{\mathbf{T}}^*_N\cdot\mathbf{u} + \bar{\mathbf{H}}^*_N\cdot\mathbf{f})\, dA + \int_{\mathfrak{P}} (\bar{\mathbf{Z}}^*\cdot\mathbf{u} + \bar{\mathbf{K}}^*\cdot\mathbf{f})\, dV - \int_{\mathfrak{P}} (\dot{\bar{\mathbf{P}}}^*\cdot\mathbf{u} + \dot{\bar{\mathbf{Q}}}^*\cdot\mathbf{f})\, dV. \tag{5.5}$$

Let $\mathbf{u}(\mathbf{X}, t)$ be an infinitesimal motion superimposed on an equilibrium configuration χ of $\mathfrak{P}$. Then upon setting $\mathbf{v} = \mathbf{u}$ in (5.4) and introducing $(4.6)_1$, we obtain the following

Work Theorem: *Let a hyperelastic body $\mathfrak{P}$ of grade 2 experience an infinitesimal generalized displacement D from an equilibrium configuration χ to a configuration χ' at time t. Then the second order increment U in the total strain energy at time t is equal to one half the work that* *would be done by the actions A over D at time t:*

$$2U = \int_{\partial\mathfrak{P}} (\mathbf{T}_N^* \cdot \mathbf{u} + \mathbf{H}_N^* \cdot \mathbf{f})\, dA + \int_{\mathfrak{P}} (\mathbf{Z}^* \cdot \mathbf{u} + \mathbf{K}^* \cdot \mathbf{f})\, dV - \int_{\mathfrak{P}} (\dot{\mathbf{P}}^* \cdot \mathbf{u} + \dot{\mathbf{Q}}^* \cdot \mathbf{f})\, dV. \tag{5.6}$$

Finally, let χ be an equilibrium configuration of $\mathfrak{P}$. Then with $\mathbf{v} = \dot{\mathbf{u}}$, (5.4) becomes

$$2\int_{\mathfrak{P}} \langle \mathbf{u}, \dot{\mathbf{u}} \rangle\, dV + \int_{\mathfrak{P}} (\dot{\mathbf{P}}^* \cdot \dot{\mathbf{u}} + \dot{\mathbf{Q}}^* \cdot \dot{\mathbf{f}})\, dV = \int_{\partial\mathfrak{P}} (\mathbf{T}_N^* \cdot \dot{\mathbf{u}} + \mathbf{H}_N^* \cdot \dot{\mathbf{f}})\, dA + \int_{\mathfrak{P}} (\mathbf{Z}^* \cdot \dot{\mathbf{u}} + \mathbf{K}^* \cdot \dot{\mathbf{f}})\, dV. \tag{5.7}$$

Also, with $\mathbf{a} = \mathbf{b} = \mathbf{u}$ in $(4.2)_1$ and (4.3), and similarly, with $\mathbf{c} = \mathbf{d} = \dot{\mathbf{u}}$ in (4.7), $(4.6)_1$ and $(4.9)_1$ yield

$$\dot{U} = \frac{d}{dt}\int_{\mathfrak{P}} \langle \mathbf{u}, \mathbf{u} \rangle\, dV = 2\int_{\mathfrak{P}} \langle \mathbf{u}, \dot{\mathbf{u}} \rangle\, dV,$$

$$\dot{K} = \frac{d}{dt}\int_{\mathfrak{P}} \langle\langle \dot{\mathbf{u}}, \dot{\mathbf{u}} \rangle\rangle\, dV = 2\int_{\mathfrak{P}} \langle\langle \dot{\mathbf{u}}, \ddot{\mathbf{u}} \rangle\rangle\, dV.$$

From (4.7) and (3.4), we find also $2\langle\langle \dot{\mathbf{u}}, \ddot{\mathbf{u}} \rangle\rangle = \dot{\mathbf{P}}^* \cdot \dot{\mathbf{u}} + \dot{\mathbf{Q}}^* \cdot \dot{\mathbf{f}}$. Thus, collecting these results in (5.7), we reach the

Principle of Incremental Energy Balance: *The time rate of change of the sum of the total incremental kinetic energy and the total incremental strain energy is equal to rate at which work is done by all surface and body actions in an incremental superimposed motion:*

$$\dot{\mathbf{K}} + \dot{\mathbf{U}} = \dot{\mathbf{W}} \equiv \int_{\partial\mathfrak{P}} (\mathbf{T}_N^* \cdot \dot{\mathbf{u}} + \mathbf{H}_N^* \cdot \dot{\mathbf{f}})\, dA + \int_{\mathfrak{P}} (\mathbf{Z}^* \cdot \dot{\mathbf{u}} + \mathbf{K}^* \cdot \dot{\mathbf{f}})\, dV. \tag{5.8}$$

Finally, integration of (5.8) over the time interval $[t_1, t_2]$, and writing $\Delta K = K(t_2) - K(t_1)$, etc., we find the

Energy Theorem: *Let a generalized displacement D corresponding to the actions A be superimposed on an equilibrium configuration of a hyperelastic material of grade 2. Then the work ΔW done by the incremental loading $\mathbf{T}_N^*$, $\mathbf{H}_N^*$, $\mathbf{Z}^*$, $\mathbf{K}^*$ equals the sum of the changes ΔK and ΔU in the second order increments of the total kinetic energy and the total strain energy for D: $\Delta W = \Delta K + \Delta U$.*

6. STABILITY OF EQUILIBRIUM UNDER DEAD LOADS

Let χ be an equilibrium configuration of $\mathfrak{P}$ subject to loading $L = \{\mathbf{T}_N, \mathbf{H}_N, \mathbf{Z}, \mathbf{K}\}$, and let $\delta\mathbf{x}(\mathbf{X}, \tau)$ be a virtual deformation corresponding to the incremental actions $A = \{\mathbf{T}_N^*, \mathbf{H}_N^*, \mathbf{Z}^*, \mathbf{K}^*\}$, τ being a suitable path parameter such that $\delta\mathbf{x}(\mathbf{X}, 0) = \mathbf{0}$. The total work w done by the total loading $L \cup A$ in the virtual deformation $\delta\mathbf{x}$ is defined by

$$w = \int_0^\tau ds \Big\{ \int_{\partial\mathfrak{P}} [(\mathbf{T}_N + \mathbf{T}_N^*) \cdot \delta\dot{\mathbf{x}} + (\mathbf{H}_N + \mathbf{H}_N^*) \cdot \nabla\delta\dot{\mathbf{x}}]\, dA + \int_{\mathfrak{P}} [(\mathbf{Z} + \mathbf{Z}^*) \cdot \delta\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}^*) \cdot \nabla\delta\dot{\mathbf{x}}]\, dV \Big\}, \tag{6.1}$$

wherein $\delta\dot{\mathbf{x}} = \partial(\delta\mathbf{x}(\mathbf{X}, s))\partial s$. The corresponding increase in the total strain energy $\mathscr{E}$ is

$$\mathscr{E} = \int_0^\tau ds \int_{\mathfrak{P}} \dot{\Sigma}(\mathbf{X}, s)\, dV = \int_{\mathfrak{P}} \Sigma^*(\mathbf{X}, \tau)\, dV \tag{6.2}$$

in which $\Sigma^* = \Sigma(\mathbf{X}, \tau) - \Sigma(\mathbf{X}, 0)$. Then, following Beatty[10, 11], we adopt the usual **Energy Criterion of Stability:** *An equilibrium configuration χ of a hyperelastic material of grade 2 is said* *to be stable for boundary conditions of place and surface actions if and only if the work done by the total loading in every virtual deformation from χ consistent with the boundary data and material constraints does not exceed the corresponding increase in the total stored energy; otherwise, it is called unstable.*

Thus, for stability we require $w \leq \mathscr{E}$ for (6.1) and (6.2) in general. In particular, when all the incremental actions A are zero, the loading L is called *dead loading*†, and the criterion becomes

$$D_E \equiv \int_{\partial \mathfrak{P}} [\mathbf{T}_N \cdot \delta\mathbf{x} + \mathbf{H}_N \cdot \nabla\delta\mathbf{x}]\, \mathrm{d}A + \int_{\mathfrak{P}} [\mathbf{Z} \cdot \delta\mathbf{x} + \mathbf{K} \cdot \nabla\delta\mathbf{x}]\, \mathrm{d}V - \int_{\mathfrak{P}} \Sigma^* \, \mathrm{d}V \leq 0. \tag{6.3}$$

Since χ is an equilibrium configuration, $\dot{\mathbf{P}}$ and $\dot{\mathbf{Q}}$ vanish in χ. Moreover, because the first two terms in (6.3) are similar in structure to those in (5.1), a parallel analysis will yield a relation similar to (5.3); thus, introducing the second order approximation (4.4)–(4.5) with $\mathbf{u} = \delta\mathbf{x}$, (6.3) yields for stability

$$D_L \equiv \int_{\mathfrak{P}} \langle \delta\mathbf{x}, \delta\mathbf{x} \rangle \, \mathrm{d}V \geq 0. \tag{6.4}$$

The following definitions are introduced for future convenience: (i) *An equilibrium configuration is said to be D_L stable for boundary conditions of place and dead load surface tractions if (6.4) holds for all infinitesimal virtual deformations that satisfy the boundary data*‡; and (ii) *An equilibrium configuration χ is called D_L superstable if χ is D_L stable with equality in (6.4) holding when and only when $\delta\mathbf{x}$ is trivially zero or corresponds to an infinitesimal, rigid virtual deformation.*§

The D_L stability criterion (6.4) is related directly to the variational principle from which the basic equations (2.2)–(2.4) were derived in [3]; namely,

$$\int_{\mathfrak{P}} (\delta L + \mathbf{Z} \cdot \delta\mathbf{x} + \mathbf{K} \cdot \delta\mathbf{F})\, \mathrm{d}V + \int_{\partial\mathfrak{P}} (\mathbf{T}_N \cdot \delta\mathbf{x} + \mathbf{H}_N \cdot \delta\mathbf{F})\, \mathrm{d}A$$

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathfrak{P}} (\mathbf{P} \cdot \delta\mathbf{x} + \mathbf{Q} \cdot \delta\mathbf{F})\, \mathrm{d}V = 0 \tag{6.5}$$

wherein $\delta\mathbf{F} = \nabla\delta\mathbf{x}$. We can establish this result by introducing a potential energy E for the equilibrium configuration χ defined by

$$E = \int_{\mathfrak{P}} (\Sigma - \mathbf{Z} \cdot \mathbf{x} - \mathbf{K} \cdot \mathbf{F})\, \mathrm{d}V - \int_{\partial\mathfrak{P}} (\mathbf{T}_N \cdot \mathbf{x} + \mathbf{H}_N \cdot \mathbf{F})\, \mathrm{d}A. \tag{6.6}$$

The increment in E due to an infinitesimal virtual displacement $\delta\mathbf{x}$ under dead loading L is given by

$$E^* = \int_{\mathfrak{P}} (\Sigma^* - \mathbf{Z} \cdot \delta\mathbf{x} - \mathbf{K} \cdot \delta\mathbf{F})\, \mathrm{d}V - \int_{\partial\mathfrak{P}} (\mathbf{T}_N \cdot \delta\mathbf{x} + \mathbf{H}_N \cdot \delta\mathbf{F})\, \mathrm{d}A. \tag{6.7}$$

Substitution of (6.7) into (6.3) shows that the exact stability functional can be written $E^* \geq 0$ for all admissible virtual deformations.

Now, $E^* = \delta E + \delta^2 E + \cdots$ and $\Sigma^* = \delta\Sigma + \delta^2\Sigma + \cdots$; hence, from (6.7),

$$\delta E = \int_{\mathfrak{P}} (\delta\Sigma - \mathbf{Z} \cdot \delta\mathbf{x} - \mathbf{K} \cdot \delta\mathbf{F})\, \mathrm{d}V - \int_{\partial\mathfrak{P}} (\mathbf{T}_N \cdot \delta\mathbf{x} + \mathbf{H}_N \cdot \delta\mathbf{F})\, \mathrm{d}A, \tag{6.8}$$

$$\delta^2 E = \int_{\mathfrak{P}} \delta^2\Sigma \, \mathrm{d}V = \int_{\mathfrak{P}} \langle \delta\mathbf{x}, \delta\mathbf{x} \rangle \, \mathrm{d}V, \tag{6.9}$$

†Of course, as remarked in [11] in a more general context, any loading for which the total work done by the incremental actions vanishes in (6.1) leads to the same form as (6.3); however, dead loading is a stronger condition requiring $A = \{\mathbf{0}\}$.

‡Material constraints will not be introduced here.

§The effect of rigid variations in a constrained Cosserat continuum is studied in [11]. We shall have no need to pursue it here.

where the last equality follows essentially from (4.4) and $(4.6)_1$. However, since χ is an equilibrium configuration, the kinetic energy function vanishes in (3.2), and we have $\dot{\mathbf{P}} \equiv \mathbf{0}$, $\dot{\mathbf{Q}} \equiv \mathbf{0}$ and $\delta L = -\delta\Sigma$ in (6.5). Hence, $\delta E = 0$ in (6.8) and $E^* = \delta^2 E$.

Of course, the vanishing of (6.8) for all virtual motions leads to the basic equations (2.2)–(2.4) that were derived from (6.5) initially. It is clear from (6.9) that the second variation in the potential energy (6.6) is equal to the second variation in the total strain energy, so we recover the stability theorem (6.4): *An equilibrium configuration χ is D_L stable (superstable) if and only if the second variation in the total strain energy is non-negative (positive) for all admissible virtual deformations from χ.*

7. THEOREMS ON MINIMUM AND MAXIMUM POTENTIAL ENERGY AND UNIQUENESS

Let χ be an equilibrium configuration of $\mathfrak{P}$ and let $\mathbf{u}(\mathbf{X})$ be a solution of the following mixed, incremental boundary value problem in the theory of small deformations superimposed on large deformations of a hyperelastic material of grade 2:

(a) *Equations of motion*:

$$\operatorname{Div} \hat{\mathbf{T}}^* + \mathbf{Z}^* = \mathbf{0} \tag{7.1}$$

wherein $\hat{\mathbf{T}}^* = \mathbf{T}^* - \operatorname{Div} \mathbf{K}^*$.

(b) *Traction boundary data*:

$$\text{(i)}\ \mathbf{T}^*_N = \mathbf{S}_N, \quad \mathbf{H}^*_N = \mathbf{M}_N \quad \text{on } \partial\mathfrak{P}_1, \tag{7.2}$$

$$\text{(ii)}\ \mathbf{H}^*_N\mathbf{N} = \mathbf{R}_N \quad \text{on } \partial\mathfrak{P}_2, \tag{7.3}$$

wherein $\partial\mathfrak{P}_1$ and $\partial\mathfrak{P}_2$ are two non-intersecting parts of $\partial\mathfrak{P}$ and $\mathbf{S}_N$, $\mathbf{M}_N$, $\mathbf{R}_N$ are certain prescribed tractions and hypertractions.

(c) *Displacement boundary data*:

$$\text{(i)}\ \mathbf{u}(\mathbf{X}, t) = \boldsymbol{\mu}(\mathbf{X}, t) \quad \text{on } \partial\mathfrak{P}\backslash\partial\mathfrak{P}_1 \equiv \partial\mathfrak{P}_3 \tag{7.4}$$

$$\text{(ii)}\ \mathbf{Du}(\mathbf{X}, t) = \boldsymbol{\nu}(\mathbf{X}, t) \quad \text{on } \partial\mathfrak{P}\backslash(\partial\mathfrak{P}_1 \cup \partial\mathfrak{P}_2) \equiv \partial\mathfrak{P}_4, \tag{7.5}$$

where $\boldsymbol{\mu}(\mathbf{X}, t)$ and $\boldsymbol{\nu}(\mathbf{X}, t)$ are prescribed functions of $\mathbf{X}$ and t on $\partial\mathfrak{P}$ as indicated. Of course, in the static case we ignore the time dependence; it is inserted here only for future convenience.

Let σ_k be the set of all non-vanishing motions that satisfy the kinematic boundary conditions (7.4)–(7.5). If $\mathbf{w} \in \sigma_k$, then $\mathbf{w}$ is called *kinematically admissible.* For *any* twice continuously differentiable displacement $\mathbf{w}$ we define an *incremental potential energy function* $P[\mathbf{w}]$ by

$$P[\mathbf{w}] \equiv \int_{\mathfrak{P}} (\langle \mathbf{w}, \mathbf{w}\rangle - \mathbf{Z}^* \cdot \mathbf{w} - \mathbf{K}^* \cdot \nabla\mathbf{w})\, \mathrm{d}V - \int_{\partial\mathfrak{P}_1} (\mathbf{S}_N \cdot \mathbf{w} + \mathbf{M}_N \cdot \nabla\mathbf{w})\, \mathrm{d}A - \int_{\partial\mathfrak{P}_2} \mathbf{R}_N \cdot \mathbf{Dw}\, \mathrm{d}A, \tag{7.6}$$

where $\langle \mathbf{w}, \mathbf{w}\rangle$ is defined by (4.1) and $\mathbf{S}_N$, $\mathbf{M}_N$, $\mathbf{R}_N$, $\mathbf{Z}^*$, $\mathbf{K}^*$ are assigned in the problem (7.1)–(7.5) for $\mathbf{u}$. Moreover, with $\mathbf{v} = \mathbf{w} - \mathbf{u}$, (4.2) yields

$$\langle \mathbf{w}, \mathbf{w}\rangle - \langle \mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle; \tag{7.7}$$

hence, the combination of (7.6) and (7.7) gives for $\mathbf{u} \in \sigma_k$

$$P[\mathbf{w}] - P[\mathbf{u}] - \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v}\rangle\, \mathrm{d}V = \int_{\mathfrak{P}} (2\langle \mathbf{u}, \mathbf{v}\rangle - \mathbf{Z}^* \cdot \mathbf{v} - \mathbf{K}^* \cdot \nabla\mathbf{v})\, \mathrm{d}V - \int_{\partial\mathfrak{P}_1} (\mathbf{S}_N \cdot \mathbf{v} + \mathbf{M}_N \cdot \nabla\mathbf{v})\, \mathrm{d}A - \int_{\partial\mathfrak{P}_2} \mathbf{R}_N \cdot D\mathbf{v}\, \mathrm{d}A. \tag{7.8}$$

Now, let $\mathbf{w} \in \sigma_k$. Then $\mathbf{v} \equiv \mathbf{0}$ and $D\mathbf{v} \equiv \mathbf{0}$ everywhere on $\partial\mathfrak{P}_3$; hence, $\mathbf{T}_N \cdot \mathbf{v} = 0$ and $\mathbf{H}_N \cdot D\mathbf{v} = 0$ on $\partial\mathfrak{P}_3$. Similarly, $\mathbf{H}_N^*\mathbf{N} \cdot D\mathbf{v} = 0$ on $\partial\mathfrak{P}_4$. In view of these conditions, $(5.2)_1$ and the traction conditions (7.2), the surface integrals in (7.8) are equivalent to $\int_{\partial\mathfrak{P}} (\mathbf{T}_N^* \cdot \mathbf{v} + \mathbf{H}_N^* \cdot \nabla\mathbf{v})\, dA$. Since $\mathbf{u}$ is a solution to the static, mixed boundary value problem (7.1)–(7.5), it satisfies the identity (5.4) with $\dot{\mathbf{P}}^* = \mathbf{0}$ and $\dot{\mathbf{Q}}^* = \mathbf{0}$. Therefore, the right hand side of (7.8) vanishes, and we have

$$P[\mathbf{w}] - P[\mathbf{u}] = \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle \, dV. \tag{7.9}$$

Consequently, if χ is D_L superstable for all virtual displacements $\mathbf{v} = \mathbf{w} - \mathbf{u}$ satisfying

$$\mathbf{v} = 0 \quad \text{on } \partial\mathfrak{P}\backslash\partial\mathfrak{P}_1, \qquad D\mathbf{v} = \mathbf{0} \quad \text{on } \partial\mathfrak{P}\backslash(\partial\mathfrak{P}_1 \cup \partial\mathfrak{P}_2), \tag{7.10}$$

we have $P[\mathbf{w}] \geq P[\mathbf{u}]$ for all $\mathbf{w} \in \sigma_k$ with equality holding when and only when $\mathbf{v}$ is an infinitesimal rigid motion or identically zero. Of course, in general the displacement boundary conditions preclude $\mathbf{v}$ being a rigid motion. We have thus established the following

Theorem on Minimum Potential Energy: *Let χ be an equilibrium configuration that is D_L superstable for all virtual deformations $\mathbf{v}$ that meet (7.10), and let $\mathbf{u}$ be a solution to the mixed, incremental boundary value problem (7.1)–(7.5). Then for all kinematically admissible deformations $\mathbf{w} \neq \mathbf{u}$ and which do not differ from $\mathbf{u}$ by an infinitesimal rigid motion, the corresponding potential energy exceeds that for $\mathbf{u}$:*

$$P[\mathbf{w}] > P[\mathbf{u}]. \tag{7.11}$$

That is, in classical terms, among all motions that satisfy the displacement boundary data and do not differ by an infinitesimal rigid or trivial motion, the one that satisfies the equilibrium equations and traction boundary conditions renders the incremental potential energy a relative minimum.

We can also derive a maximum potential energy principle. Let σ_t be the set of all non-vanishing motions that satisfy the equations of equilibrium and the traction boundary conditions (7.1)–(7.3). If $\mathbf{w} \in \sigma_t$, then $\mathbf{w}$ is said to be *statically admissible.* Now, for *any* $\mathbf{w}$ we define the *incremental complementary energy functional* $Q[\mathbf{w}]$ by

$$Q[\mathbf{w}] \equiv P[\mathbf{w}] + \int_{\partial\mathfrak{P}_3} [\mathbf{T}_N^* \cdot (\boldsymbol{\mu} - \mathbf{w}) + \mathbf{H}_N^* \cdot \nabla(\boldsymbol{\mu} - \mathbf{w})]\, dA + \int_{\partial\mathfrak{P}_4} \mathbf{H}_N^*\mathbf{N} \cdot (\boldsymbol{\nu} - D\mathbf{w})\, dA \tag{7.12}$$

in which $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ are assigned in the problem (7.1)–(7.5) for a solution $\mathbf{u}$. Hence, $Q[\mathbf{u}] = P[\mathbf{u}]$. We may interchange $\mathbf{u}$ and $\mathbf{w}$ in (7.7) to obtain

$$\langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle,$$

where $\mathbf{v} = \mathbf{u} - \mathbf{w}$: Then with (7.6) we find

$$P[\mathbf{u}] - \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle \, dV = P[\mathbf{w}] + \int_{\mathfrak{P}} [2\langle \mathbf{v}, \mathbf{w} \rangle - \mathbf{Z}^* \cdot \mathbf{v} - \mathbf{K}^* \cdot \nabla\mathbf{v}]$$
$$- \int_{\partial\mathfrak{P}_1} [\mathbf{S}_N \cdot \mathbf{v} + \mathbf{M}_N \cdot \nabla\mathbf{v}]\, dA - \int_{\partial\mathfrak{P}_2} \mathbf{R}_N \cdot D\mathbf{v}\, dA, \tag{7.13}$$

wherein $\mathbf{Z}^*$, $\mathbf{K}^*$, $\mathbf{S}_N$, $\mathbf{M}_N$, $\mathbf{R}_N$ are assigned in the problem (7.1)–(7.5) for $\mathbf{u}$.

Let $\mathbf{w} \in \sigma_t$, and put $\dot{\mathbf{P}}^* = \mathbf{0}$ and $\dot{\mathbf{Q}}^* = \mathbf{0}$ in (5.4). Then application of the traction boundary conditions (7.2)–(7.3) and the rule (5.2) shows that the right hand side of (7.13) is the same as that in (7.12); therefore, we have

$$Q[\mathbf{u}] - Q[\mathbf{w}] = \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle \, dV. \tag{7.14}$$

Thus, if χ is D_L superstable for all virtual deformations $\mathbf{v} = \mathbf{u} - \mathbf{w}$ that satisfy the null equations

$$\text{Div}\,\mathbf{T}^* = \mathbf{0} \text{ in } \mathfrak{P}; \quad \mathbf{T}_N^* = \mathbf{0},\ \mathbf{H}_N^* = \mathbf{0} \text{ on } \partial\mathfrak{P}_1; \quad \mathbf{H}_N^*\mathbf{N} = \mathbf{0} \text{ on } \partial\mathfrak{P}_2, \tag{7.15}$$

then $Q[\mathbf{u}] \geq Q[\mathbf{w}]$ for all $\mathbf{w} \in \sigma_t$, the equality holding when and only when $\mathbf{v}$ is an infinitesimal rigid motion or identically zero. We have proved the following

Theorem on Maximum Potential Energy: *If an equilibrium configuration is D_L superstable for all virtual motions $\mathbf{v} = \mathbf{u} - \mathbf{w}$ that satisfy* (7.15) *and for which* $\mathbf{u}$ *is a solution to the mixed, incremental boundary value problem* (7.1)–(7.5), *for all statically admissible deformations* $\mathbf{w} \neq \mathbf{u}$ *and which do not differ from* $\mathbf{u}$ *by an infinitesimal rigid motion, the corresponding complementary energy is less than that for* $\mathbf{U}$:

$$Q[\mathbf{u}] > Q[\mathbf{w}]. \tag{7.16}$$

That is, among all motions that satisfy the equilibrium equations and traction boundary data, and which do not differ by an infinitesimal rigid or trivial motion, the one that satisfies also the displacement boundary data renders the complementary potential energy a relative maximum.

Finally, let us assume that $\mathbf{u}$ and $\mathbf{w}$ are distinct solutions of the mixed, incremental boundary value problem and do not differ by an infinitesimal rigid body motion; and assume also that χ is D_L superstable for $\mathbf{v} = \mathbf{w} - \mathbf{u}$ that satisfies (7.10), or for $\mathbf{v} = \mathbf{u} - \mathbf{w}$ that satisfies (7.15). Then from (7.9), since both $\mathbf{u}$ and $\mathbf{w}$ are solutions of the boundary value problem,

$$P[\mathbf{w}] - P[\mathbf{u}] = \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle \, \mathrm{d}V = \int_{\mathfrak{P}} \langle -\mathbf{v}, -\mathbf{v} \rangle \, \mathrm{d}V = P[\mathbf{u}] - P[\mathbf{w}],$$

that is, $P[\mathbf{w}] = P[\mathbf{u}]$. Similarly, $Q[\mathbf{w}] = Q[\mathbf{u}]$ from (7.14). However, these violate (7.11) and (7.16) unless $\mathbf{w} = \mathbf{u}$. We have the following

Uniqueness Theorem of Equilibrium: *If χ is an equilibrium configuration that is D_L superstable for all virtual displacements that satisfy* (7.10) *or* (7.15) *and which do not differ by a rigid body motion, then the mixed, incremental boundary value problem* (7.1)–(7.5) *has at most one solution* $\mathbf{u}(\mathbf{X})$; *and this is the solution for which the potential energy is a relative minimum for all kinematically admissible displacements and for which the complementary energy is a relative maximum for all statically admissible displacements.*

In the traction boundary value problem $\partial\mathfrak{P}_1 = \partial\mathfrak{P}$, $\partial\mathfrak{P}_2 = \phi$, the empty set, and the class of admissible motions is the set of all infinitesimal deformations. In this case $\mathbf{v}$ may be a uniform rigid body motion; and all motions $\mathbf{v} = \mathbf{a}$, constant, and some motions [see Ref. 11] that differ by a rigid body rotation are admissible and have the same potential and complementary energies as the solution $\mathbf{u}$. This means that the solution to the traction problem is unique to within an infinitesimal rigid motion.

Finally, if χ is only D_L stable instead of superstable, there exists some class of virtual deformations $\sigma_0 \subset \sigma_k$ such that for all non-trivial $\mathbf{v} \in \sigma_0$, $\mathbf{v}$ satisfies (7.10) and $\int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle \, \mathrm{d}V = 0$. When this is so, and if for all other $\mathbf{v} \in \sigma_0$ we have $\int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle \, \mathrm{d}V > 0$, then the solution $\mathbf{u}(\mathbf{X})$ to the mixed problem is, at most, unique to within a motion belonging to the class σ_0; and the equilibrium configuration is said to be neutrally stable.

8. UNIQUENES THEOREM FOR THE MIXED, DYNAMIC BOUNDARY VALUE PROBLEM

The mixed, incremental boundary-initial value problem of motion consists in finding the time dependent displacement field $\mathbf{u}(\mathbf{X}, t)$ that satisfies the incremental equations of motion (3.8) for assigned body and hyper-body forces together with the traction and time dependent displacement data (7.2)–(7.5), and the initial conditions at $t = 0$ in χ: $\mathbf{u}(\mathbf{X}, 0) = \mathbf{a}(\mathbf{X})$, $\dot{\mathbf{u}}(\mathbf{X}, 0) = \mathbf{b}(\mathbf{X})$.

To show for this problem that two solutions $\mathbf{u}_1(\mathbf{X}, t)$ and $\mathbf{u}_2(\mathbf{X}, t)$ are the same, it suffices to show that their difference $\mathbf{v}(\mathbf{X}, t) = \mathbf{u}_1(\mathbf{X}, t) - \mathbf{u}_2(\mathbf{X}, t)$ is identically zero for all times t; $\mathbf{v}(\mathbf{X}, t)$ being the solution to the following difference problem for dead load body and hyper-body forces:

(a) *Equations of motion*:

$$\mathrm{Div}\,\mathbf{T}^* = \dot{\mathbf{P}}^* - \mathrm{Div}\,\dot{Q}^*. \tag{8.1}$$

(b) *Traction boundary data*:

$$\text{(i) } \mathbf{T}^*_N = \mathbf{0},\ \mathbf{H}^*_N = \mathbf{0} \text{ on } \partial\mathfrak{P}_1 \text{ of } \partial\mathfrak{P} \tag{8.2}$$

$$\text{(ii) } \mathbf{H}^*_N\mathbf{N} = \mathbf{0} \text{ on } \partial\mathfrak{P}_2 \text{ of } \partial\mathfrak{P} \text{ such that } \partial\mathfrak{P}_1 \cap \partial\mathfrak{P}_2 = \phi, \tag{8.3}$$

(c) *Displacement boundary data*:

$$\text{(i) } \mathbf{v}(\mathbf{X}, t) = \mathbf{0} \text{ on } \partial\mathfrak{P}\backslash\partial\mathfrak{P}_1\ \forall\, t \geq 0, \tag{8.4}$$

which implies also that $\mathbf{Dv} = \mathbf{0}$ on $\partial\mathfrak{P}\backslash\partial\mathfrak{P}_1\ \forall\, t \geq 0$, and

$$\text{(ii) } D\mathbf{v}(\mathbf{X}, t) = \mathbf{0} \text{ on } \partial\mathfrak{P}\backslash(\partial\mathfrak{P}_1 \cup \partial\mathfrak{P}_2) \tag{8.5}$$

(d) *Initial conditions*:

$$\text{(i) } \mathbf{v}(\mathbf{X}, 0) = \mathbf{0},$$

$$\text{(ii) } \dot{\mathbf{v}}(\mathbf{X}, 0) = \mathbf{0} \text{ everywhere initially.} \tag{8.6}$$

Since $\mathbf{v}$ is a solution of the incremental equations of motion with null incremental body and hyper-body forces, our energy theorem (5.8) implies

$$\int_{\partial\mathfrak{P}} (\mathbf{T}^*_N \cdot \dot{\mathbf{v}} + \mathbf{H}^*_N \cdot \nabla\dot{\mathbf{v}})\, \mathrm{d}A = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathfrak{P}} \langle\langle \dot{\mathbf{v}}, \dot{\mathbf{v}} \rangle\rangle\, \mathrm{d}V + \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle\, \mathrm{d}V\right). \tag{8.7}$$

With the aid of (5.2) and application of the boundary data (8.2)–(8.5), it can be shown that the surface integral in (8.7) vanishes for all times $t \geq 0$. Hence, (8.6) and (8.7) yield for all times t

$$\int_{\mathfrak{P}} \langle\langle \dot{\mathbf{v}}, \dot{\mathbf{v}} \rangle\rangle\, \mathrm{d}V + \int_{\mathfrak{P}} \langle \mathbf{v}, \mathbf{v} \rangle\, \mathrm{d}V = 0. \tag{8.8}$$

It is easy to see from (4.8) that for the special case (3.7) the kinetic energy $K = \int_{\mathfrak{P}} (\frac{1}{2}\rho_\kappa \dot{\mathbf{v}} \cdot \dot{\mathbf{v}} + \frac{1}{2}\mathbf{I}\nabla\dot{\mathbf{v}} \cdot \nabla\dot{\mathbf{v}})\, \mathrm{d}V \geq 0$, equality holding only when $\dot{\mathbf{v}} = \mathbf{0}$. Thus, let us assume henceforward that the following kinetic energy response functional is positive definite, namely,

$$\int_{\mathfrak{P}} \langle\langle \mathbf{a}, \mathbf{a} \rangle\rangle\, \mathrm{d}V > 0 \tag{8.9}$$

for all vectors $\mathbf{a} \neq \mathbf{0}$. Then $K = \int_{\mathfrak{P}} \langle\langle \dot{\mathbf{v}}, \dot{\mathbf{v}} \rangle\rangle\, \mathrm{d}V \geq 0$, vanishing for all times if and only if $\dot{\mathbf{v}} = \mathbf{0}$. Now, if χ is D_L stable for all motions $\mathbf{v}(\mathbf{X}, t)$ that satisfy (8.1)–(8.6), equation (8.8) implies that each integral must vanish separately. Hence, $\dot{\mathbf{v}}(\mathbf{X}, t) = \mathbf{0}$; that is, for all times, $\mathbf{v}(\mathbf{X}, t) = \mathbf{v}(\mathbf{X}, 0) = \mathbf{0}$ by (8.6). We thus reach our

Uniqueness Theorem of Dynamics: *Suppose a configuration χ in equilibrium under dead load body actions is D_L stable for all motions that meet* (8.1)–(8.6) *and that the kinetic energy functional is positive definite. Then the mixed incremental boundary-initial value problem for a small superimposed motion has at most one solution. Moreover, the displacement problem for which $\partial\mathfrak{P}_1 = \partial\mathfrak{P}_2 = \phi$, and the traction problem for which $\partial\mathfrak{P}_1 = \partial\mathfrak{P}$, also have unique solutions.*

9. NORMAL MODES AND STABILITY

Let p denote the constant circular frequency of some normal mode motion and consider an incremental motion $\tilde{\mathbf{u}}(\mathbf{X}, t) = \tilde{\mathbf{u}}(\mathbf{X})\varphi(t)$ in which $\tilde{\mathbf{u}}(\mathbf{X})$ is a continuously differentiable vector valued function of $\mathbf{X}$ alone and $\varphi(t) = e^{ipt}$. It is clear from (4.1) that $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \tilde{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \varphi(2t)$. For any tensor valued function $\mathbf{Y}$ that is linear in $\mathbf{u}$ or its gradients (3.1), we write $\tilde{\mathbf{Y}} = \mathbf{Y}(\tilde{\mathbf{u}})$. Then $\mathbf{Y}(\mathbf{u}) = \mathbf{Y}(\tilde{\mathbf{u}})\varphi(t) = \tilde{\mathbf{Y}}\varphi(t)$; and by (3.3) and (3.4) we have $\mathbf{T}^* = \tilde{\mathbf{T}}^*\varphi(t)$, $\mathbf{H}^* = \tilde{\mathbf{H}}^*\varphi(t)$, $\mathbf{P}^* = ip\tilde{\mathbf{P}}^*\varphi(t)$ and $\mathbf{Q}^* = ip\tilde{\mathbf{Q}}^*\varphi(t)$. Equations (3.8)–(3.10) for dead load body actions thus provide the following equations of motion and traction boundary conditions:

$$\operatorname{Div} \tilde{\mathbf{T}}^* + p^2(\tilde{\mathbf{P}}^* - \operatorname{Div} \tilde{\mathbf{Q}}^*) = \mathbf{0}, \tag{9.1}$$

$$\mathbf{T}^*_N - \mathbf{D} \cdot \mathbf{H}^*_N = [\tilde{\mathbf{T}}^*\mathbf{N} - \mathbf{D} \cdot (\tilde{\mathbf{H}}^*\mathbf{N}) - p^2\tilde{\mathbf{Q}}^*\mathbf{N}]\varphi(t) \tag{9.2}$$

$$\mathbf{H}^*_N\mathbf{N} = \tilde{\mathbf{H}}^* \cdot (\mathbf{N} \otimes \mathbf{N})\varphi(t), \quad [\mathbf{H}^*_N\mathbf{M}] = [\tilde{\mathbf{H}}^* \cdot (\mathbf{M} \otimes \mathbf{N})]\varphi(t). \tag{9.3}$$

Permissible values of p are determined by the specified traction boundary conditions (9.2)–(9.3) when the solution field $\tilde{\mathbf{u}}$ determined by (9.1) is inserted into these. The resultant equation for p is called the frequency equation and its roots are called normal frequencies. The normal mode of frequency p_k is given by $\mathbf{u}(\mathbf{X}, t) = \mathbf{u}_{\underline{k}}(\mathbf{X})\varphi_{\underline{k}}(t)$, where† $\varphi_{\underline{k}}(t) = A_{\underline{k}}e^{ip_{\underline{k}}t}$, A_k being an arbitrary constant multiplier and $\mathbf{u}_k(\mathbf{X})$ the normal mode function corresponding to p_k. The general incremental motion of $\mathfrak{P}$ is assumed to be a superposition of these normal mode motions:

$$\mathbf{u}(\mathbf{X}, t) = \sum_{k=1}^{\infty} \mathbf{u}_k(\mathbf{X})\varphi_k(t).$$

We shall prove the following

Theorem on Orthogonality of Normal Mode Functions: *Let the body actions be constant fields so that* $\mathbf{Z}^* = \mathbf{0}$, $\mathbf{K}^* = \mathbf{0}$, *and let the boundary conditions* (9.2)–(9.3) *satisfy*

$$\int_{\partial\mathfrak{P}} (\mathbf{u}_k \cdot \hat{\mathbf{T}}_l{}^* + \mathbf{f}_k \cdot \mathbf{H}_l{}^*)\mathbf{N}\, dA = \int_{\partial\mathfrak{P}} (\mathbf{u}_l \cdot \hat{\mathbf{T}}^*_k + \mathbf{f}_l \cdot \mathbf{H}^*_k)\mathbf{N}\, dA, \tag{9.4}$$

where $\hat{\mathbf{T}}_l{}^* \equiv \hat{\mathbf{T}}^*(\mathbf{u}_{\underline{l}}\varphi_{\underline{l}}) = \hat{\mathbf{T}}_{\underline{l}}{}^*(\mathbf{u}_{\underline{l}})\varphi_l$, *etc., and* $\mathbf{f}_k = \nabla\mathbf{u}_k$, *for any pair of distinct normal motions having distinct, squared normal frequencies* $p_k{}^2$ *and* $p_l{}^2$. *Then the normal mode functions are orthogonal*; *i.e.*

$$\int_{\mathfrak{P}} \langle\langle \mathbf{u}_k, \mathbf{u}_l \rangle\rangle\, dV = 0. \tag{9.5}$$

Proof: For non-trivial fields $\mathbf{u} = \mathbf{u}_{\underline{k}}\varphi_{\underline{k}}$, $\bar{\mathbf{u}} = \mathbf{u}_l\varphi_{\underline{l}}$ the reciprocal theorem (5.5) together with (4.8) and the commutative property of (4.7) yields

$$\int_{\partial\mathfrak{P}} (\mathbf{T}^*_N \cdot \bar{\mathbf{u}} + \mathbf{H}^*_N \cdot \bar{\mathbf{f}})\, dA - \int_{\partial\mathfrak{P}} (\bar{\mathbf{T}}^*_N \cdot \mathbf{u} + \bar{\mathbf{H}}^*_N \cdot \mathbf{f})\, dA = 2(p_{\underline{l}}^2 - p_{\underline{k}}^2)\varphi_{\underline{k}}\varphi_{\underline{l}} \int_{\mathfrak{P}} \langle\langle \mathbf{u}_k, \mathbf{u}_l \rangle\rangle\, dV,$$

where $\mathbf{T}^*_N$, $\mathbf{H}^*_N$ and $\bar{\mathbf{T}}^*_N$, $\bar{\mathbf{H}}^*_N$ denote the surface actions corresponding to the incremental vibrations $\mathbf{u}$ and $\bar{\mathbf{u}}$, respectively. Now, with the boundary conditions (9.2)–(9.3) and by the same kind of analysis that led to (5.3), it can be shown that

$$\int_{\partial\mathfrak{P}} (\mathbf{T}^*_N \cdot \bar{\mathbf{u}} + \mathbf{H}^*_N \cdot \bar{\mathbf{f}})\, dA = \varphi_{\underline{l}}\varphi_{\underline{k}} \int_{\partial\mathfrak{P}} (\mathbf{u}_l \cdot \hat{\mathbf{T}}^*_k + \mathbf{f}_l \cdot \mathbf{H}^*_k)\mathbf{N}\, dA.$$

Therefore, if (9.4) holds, we shall have

$$(p_l^2 - p_k^2) \int_{\mathfrak{P}} \langle\langle \mathbf{u}_{\underline{k}}, \mathbf{u}_{\underline{l}} \rangle\rangle\, dV = 0. \tag{9.6}$$

Thus, for all $p_k{}^2 \neq p_l{}^2$, (9.5) follows from (9.6); and the proof is finished.

†A bar beneath a repeated index indicates no sum on k.

In the special case (3.7), we find for (9.5)

$$\int_{\mathfrak{P}} \langle\langle \mathbf{u}_k, \mathbf{u}_l \rangle\rangle \, dV = \int_{\mathfrak{P}} (\rho_\kappa \mathbf{u}_k \cdot \mathbf{u}_l + \mathbf{I}\mathbf{f}_k \cdot \mathbf{f}_l) \, dV = 0;$$

and when $\mathbf{I} = \mathbf{0}$, we recover the familiar orthogonality condition. The result (9.5) and the constraint (8.9) are required for the

Frequency Theorem: *Consider a small vibration superimposed upon an equilibrium configuration that is D_L superstable under dead load body actions and assume that the kinetic energy response functional is positive definite. Then, if*

$$\int_{\partial \mathfrak{P}} (\mathbf{u}_{\underline{k}} \cdot \tilde{\mathbf{T}}^{\dagger}_{\underline{k}} + \mathbf{f}_{\underline{k}} \cdot \mathbf{H}^{\dagger}_{\underline{k}}) \mathbf{N} \, dA = 0 \quad \textit{for each } \mathbf{u}_k, \tag{9.7}$$

all roots of the frequency equation are real and non-vanishing.

Proof: If the frequency equation has an imaginary root $p_k^2 = a + ib$, then $p_l^2 = a - ib$ is also a root. Since the corresponding normal mode functions $\mathbf{u}_k$ and $\mathbf{u}_l$ are complex conjugate functions, then with (8.9) it can be proved that $\int_{\mathfrak{P}} \langle\langle \mathbf{u}_k, \mathbf{u}_l \rangle\rangle \, dV > 0$. But this contradicts (9.6) unless $p_l^2 = p_k^2$, i.e. unless $b = 0$. It remains to show that $a > 0$.

Using $\mathbf{u} = \mathbf{u}_{\underline{k}} \varphi_{\underline{k}}$ in the energy principle (5.8), we see easily that whenever (9.1) holds and the incremental body actions are zero,

$$p_{\underline{k}}^2 \int_{\mathfrak{P}} \langle\langle \mathbf{u}_{\underline{k}}, \mathbf{u}_{\underline{k}} \rangle\rangle \, dV = \int_{\mathfrak{P}} \langle \mathbf{u}_{\underline{k}}, \mathbf{u}_{\underline{k}} \rangle \, dV. \tag{9.8}$$

Thus, if χ is D_L superstable and (8.9) holds, then $p_k^2 > 0$ follows.

We note that if χ is only D_L stable, then for some $\mathbf{u}_k \neq \mathbf{0}$ it is possible that $\langle \mathbf{u}_{\underline{k}}, \mathbf{u}_{\underline{k}} \rangle = 0$; hence, $p_k^2 = 0$. Also, if $a < 0$ for some k, $p_k^2 < 0$ and (9.8) implies that the equilibrium configuration χ is unstable, i.e. $\int_{\mathfrak{P}} \langle \mathbf{u}_{\underline{k}}, \mathbf{u}_{\underline{k}} \rangle \, dV < 0$ unless $\mathbf{u} \equiv \mathbf{0}$.

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